

Eluding SUSY at every genus on stable closed string vacua.

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ABSTRACT: In closed string vacua, ergodicity of unipotent flows provide a key for relating vacuum stability to the UV behavior of spectra and interactions. Infrared finiteness *at all genera* in perturbation theory can be rephrased in terms of cancelations involving *only tree-level* closed strings scattering amplitudes. This provides quantitative results on the allowed deviations from supersymmetry on perturbative stable vacua. From a mathematical perspective, diagrammatic relations involving closed string amplitudes suggest a relevance of unipotent flows dynamics for the Schottky problem and for the construction of the superstring measure.

KEYWORDS: Closed String Amplitudes, Vacuum Stability, SUSY Breaking, Homogenous Dynamics, Uniform Distribution, Unipotent Flows, Riemann Hypothesis, Schottky Problem.

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1. Introduction

In recent years, mathematical results in homogenous space dynamics have been leading to striking results in number theory, (see for example [EW],[EMV],[ELMV]). In a series of papers [C1],[C2],[CC1],[CC2],[ACER], we have proved that fruitful interactions arise also between homogenous space dynamics and string theory. This research lines lead to results both in string theory [C1],[CC1],[ACER], and in the mathematics of the automorphic forms and of unipotent flows in homogenous spaces [C2],[CC2].

In a well known paper, Kutasov and Seiberg [KS] have shown that in backgrounds with no tachyons, closed string spectra exhibit a global UV asymptotic Fermi-Bose degeneracy. They dubbed this global cancelation among bosonic and fermionic degrees of freedom as *Asymptotic Supersymmetry*. This UV property of closed string spectra is related by modularity to infrared finiteness of the one-loop amplitude. At genus one level, it indicates allowed deviations from supersymmetry on a stable vacuum. It was remarked in [C1] that this UV property of (one-loop) stable closed string vacua is related to mathematical theorems on uniform distribution of long horocycles in the modular surface $SL(2, \mathbb{Z}) \backslash SL(2, \mathbb{R})$.

It is the purpose of this paper to extend the original one-loop (genus one) analysis of [KS] to all genera in closed string perturbation theory. We achieve this goal by using certain mathematical theorems on the dynamics of multidimensional unipotent flows, we have recently obtained in [CC2]. Once applied to genus g closed string amplitudes, those

results translate into relatively simple operations at a diagrammatic level. They correspond to suitable cuts of a Riemann surface handle(s). This allows to rewrite a genus g vacuum amplitude with g handles as a sum of genus $(g - 1)$ two-point functions. Uniformization results in [CC2] instruct to sum over all physical string states flowing through the $(g - 1)$ amplitude two-external legs, (figure 8). The sum over physical states is regulated by an ultraviolet cutoff Λ , and uniform distribution theorem, (Theorem 2 in [CC2]), ensures that the original vacuum amplitude is recovered in the $\Lambda \rightarrow \infty$ limit. Moreover, for large Λ one has that the error term is under control [CC2]. Interestingly, the error estimate is intimately related to the non trivial zeros of the Riemann zeta function, and a result for this quantity would prove or disprove the Riemann hypothesis [CC1],[CC2].

In [CC2] was proved, (Theorem 1), that the unipotent average of the string integrand automorphic function is a modular invariant function under the genus $(g - 1)$ modular group. This is a crucial property, in order to be able to apply iteratively the uniform distribution theorem. In a diagrammatic language this corresponds to cutting handles in a closed string vacuum amplitude, and transmute each handle into a pair of external legs. In this way, one is able to reduce a genus g vacuum amplitude into a sum over tree-level amplitudes with $2g$ external legs, (see again figure 8). Infrared finiteness of the genus g vacuum amplitude is then translated in constraints involving ultraviolet cancelations among $2g$ -point *tree level* amplitudes. This is the way closed strings can elude SUSY on stable vacua. In a sense our results provide the completion *at all genera* of the condition of *Asymptotic Supersymmetry*, obtained at one-loop level in [KS], (see also [Di] for related work).

2. Genus one: uniform distribution of long horocycles viz cutting the torus handle

The one-loop torus amplitude is given by the following modular integral

$$A_1 = \int_{\mathcal{D}_1} dw dv v^{-\frac{d}{2}-1} \text{Str} \left(e^{2\pi i w (L_0 - \bar{L}_0)} e^{-\pi v (L_0 + \bar{L}_0)} \right), \quad (2.1)$$

where $\tau = w + iv$, is the worldsheet torus modulus, $w \in \mathbb{R}$, $v > 0$. As we shall see, this notation for the real and imaginary parts of τ reflects Iwasawa coordinatization of the genus $g = 1$ upper complex plane \mathcal{H}_1 . In (2.1) d is the number of non compact space-time directions, the supertrace assigns a minus sign to fermionic closed string states, L_0 and \bar{L}_0 are the zero modes of the Virasoro operators, and the integral is performed on a modular domain $\mathcal{D}_1 \sim SL(2, \mathbb{Z}) \backslash \mathcal{H}_1$, with \mathcal{H}_1 the upper complex plane.

On the subregion of \mathcal{D}_1 , where $v > 1$, (see figure 1), the w coordinate is integrated mod(1). This integration enforces the physical condition of level matching $(L_0 - \bar{L}_0)|\Phi\rangle = 0$, which selects closed string *physical* states. The one-loop torus vacuum amplitude A_1 in the representation given in (2.1) receives contributions from *non physical* closed string states,

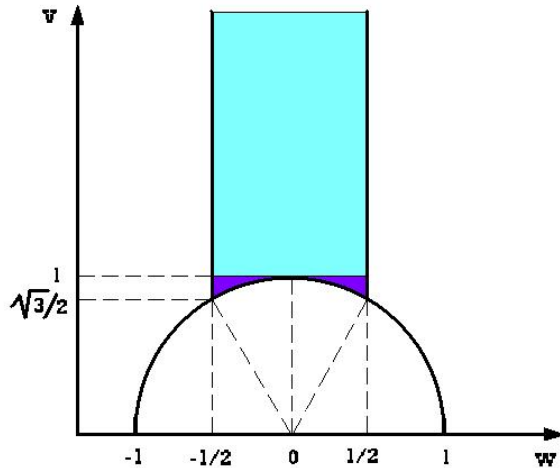


Figure 1: The torus modular surface in the upper complex plane. On the shaded integration region below $v = 1$, non-physical closed string states do contribute to the one-loop vacuum energy.

from the integration subregion where $\sqrt{3}/2 < v < 1$, (the shaded region in figure 1). Since the integration region \mathcal{D}_1 does not touch the \mathcal{H}_1 boundary $v \rightarrow 0$, the UV physical region is not probed, and the vacuum amplitude is free from ultraviolet problems.

However, there is an alternative representation for the genus one vacuum amplitude A_1 , which allows to probe the ultraviolet properties of the closed string spectrum. In this representation, one can check that A_1 receives contribution *only* from physical states. This alternative description [C1] follows by using ergodic properties of the horocycle flow [He],[Fu],[DS], (figure 2). Ergodicity of the horocycle flow states that the modular image of the horocycle $H_\alpha = \mathbb{R} + i\alpha \subset \mathcal{H}_1$, in the $\alpha \rightarrow 0$ limit covers uniformly the modular domain \mathcal{D}_1 . This implies that for a continuous bounded modular function $f = f(w, v)$ its horocycle average $\langle f \rangle_{H_\alpha}$ for $\alpha \rightarrow 0$ tends to its average on the modular region $\langle f \rangle_{\mathcal{D}_1}$

$$\lim_{\alpha \rightarrow 0} \langle f \rangle_{H_\alpha} = \lim_{v \rightarrow 0} \int_0^1 dw f(w, v) = \frac{1}{\text{Vol}(\mathcal{D}_1)} \int_{\mathcal{D}_1} dw dv v^{-2} f(w, v) = \langle f \rangle_{\mathcal{D}_1}, \quad (2.2)$$

where $\text{Vol}(\mathcal{D}_1) = \pi/3$. Notice, that the l.h.s. is indeed the f average along H_v computed with the \mathcal{H}_1 hyperbolic metric $ds^2 = v^{-2}(dw^2 + dv^2)$ ¹.

¹Equation (2.2) holds for every continuous bounded modular function f , however, in string theory, in the absence of tachyons, generically one has to deal with modular functions of polynomial growth at infinity (type II theories), or of exponential growth at infinity (Heterotic theories). In this latter case, the modular invariant integrand function contains terms of exponential growth for $v \rightarrow \infty$, which are removed by w integration mod (1). This terms are dubbed as *unphysical tachyons* [KS], since they correspond to tachyonic states in the supertrace (2.1) that do not respect level matching. In the type II case, eq. (2.2) is proved to hold [Za2],[ACER],[C2], while in the heterotic case eq. (2.2) is expected to hold [ACER],[C2] on physical grounds, although this has not been actually proved, (see [C2] for a discussion and some related mathematical results on this problem)

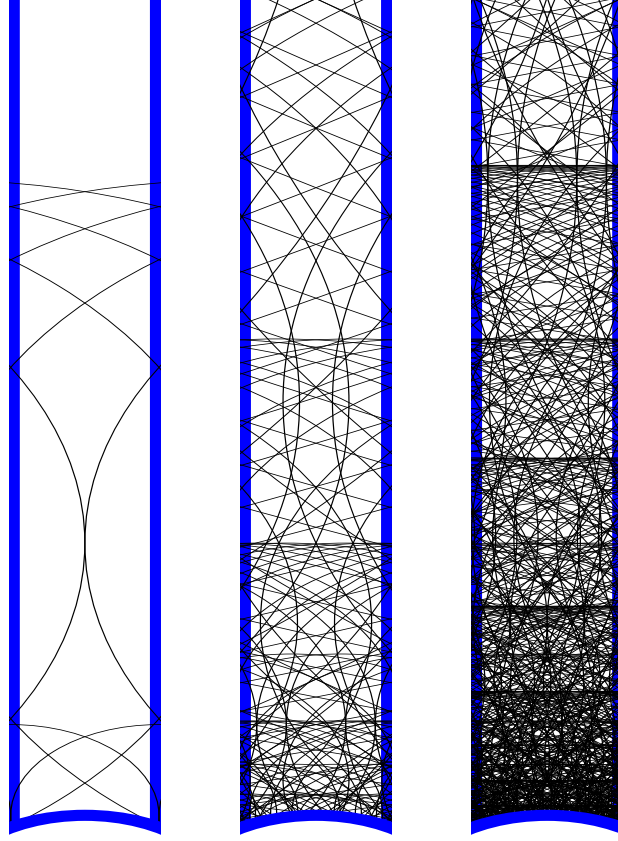


Figure 2: Modular images of horocycles of increasing length. In the upper complex plane the horizontal line $H_\alpha := \mathbb{R} + i\alpha$ is called a horocycle, since it can be thought as a circle tangent to infinity. H_α is modularly equivalent to a (infinite countable) family of circles, all tangent to the real axis in rational points. It is interesting to map H_α in the standard fundamental domain and observe the behavior of its modular image in the $\alpha \rightarrow 0$ limit. Due to the subgroup Γ_∞ of the modular group $SL(2, \mathbb{Z})$ given by integral translations along the real axis, it is enough to map in the modular domain the segment $\Gamma_\infty \backslash H_\alpha = [-1/2, 1/2) + i\alpha$, with hyperbolic length $1/\alpha$. We refer to this latter quantity as the length of the horocycle H_α . What happens to the modular image in the standard modular domain of the horocycle in the increasing length limit $\alpha \rightarrow 0$ can be observed in figure. Left: modular image of the line $y = \frac{1}{8}$. Center: modular image of the line $y = \frac{1}{100}$. Right: modular image of the line $y = \frac{1}{400}$. In all cases the modular domain is truncated up to a $y \leq 10$. The modular image of a line $y = \alpha$ tends to become dense in the modular domain as the horizontal line gets close to the real axis, ($\alpha \rightarrow 0$ limit). Indeed, in the $\alpha \rightarrow 0$ limit the modular image of the horocycle $y = \alpha$ tends to uniformly cover the modular domain [He].

By applying the uniform distribution result (2.2) to the genus one torus amplitude A_1 (2.1), one finds

$$A_1 = Vol(\mathcal{D}_1) \lim_{v \rightarrow 0} v^{1-d/2} Str \left(e^{-\pi v(L_0 + \bar{L}_0)} \int_{mod(1)} dw e^{2\pi i w(L_0 - \bar{L}_0)} \right). \quad (2.3)$$

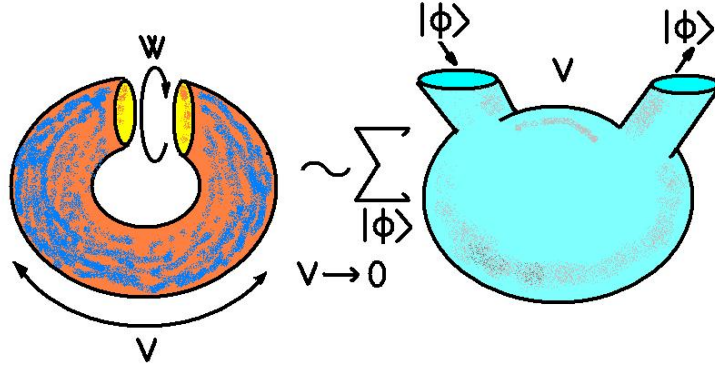


Figure 3: The uniform distribution theorem applied to the one-loop vacuum amplitude (on the left), gives an ultraviolet representation for this vacuum amplitude in terms of a sum over physical states of tree level (sphere) two-points amplitudes. Diagrammatically this relation prescribes to cut the torus handle thus creating a sphere with two marking points. Uniform distribution theorem instructs to sum over physical closed string states through the two marked points.

This latter quantity has an enumerative meaning related to the towers of massive closed string excitations. In terms of effective numbers of closed string states reads

$$A_1 = Vol(\mathcal{D}_1) \lim_{v \rightarrow 0} v^{1-d/2} \sum_{|\Phi\rangle} (-)^{F_\Phi} d(\Phi) e^{-\pi v m_\Phi^2}, \quad (2.4)$$

where the sum is restricted to closed string *physical* states $|\Phi\rangle$, $(L_0 - \bar{L}_0)|\Phi\rangle = 0$. Fermionic states are counted with a minus sign, and $d(\Phi)$ is the number of physical polarizations of $|\Phi\rangle$ of mass m_Φ . At a diagrammatical level, the above form suggests that one can cut the torus handle, and obtain an equivalent representation of the genus one vacuum amplitude as a sum of tree level two-points amplitudes restricted to *physical* closed string states, (this is illustrated in figure 3).

One-loop vacuum stability, (absence of closed string tachyons), corresponds to the finiteness of the vacuum amplitude A_1 . This implies via eq. (2.4) a constraint on the allowed deviation from supersymmetry of one-loop stable closed string vacua. Convergence of the series of tree-level two-point functions in (2.4) requires an overall Fermi-Bose degeneracy of the string spectrum. Let us notice the role of the $v > 0$ coordinate, (the imaginary part of the torus modulus τ), as a ultraviolet cutoff $\Lambda_{uv} = 1/v$ for the mass of the states contributing to eq. (2.4). Equality in eq. (2.4) holds in the ultraviolet limit $v \rightarrow 0$, however one can also consider this relation for small $v > 0$. It turns out that the error term is under control, and it goes to zero *polynomially* in the $v \rightarrow 0$ limit. A remarkable fact is that the vanishing rate in the error estimate is intimately connected to the Riemann hypothesis. This was discovered for modular functions $f = f(w, v)$ of rapid decay by Zagier [Za1]:

$$\int_0^1 dw f(w, v) \sim \frac{1}{\text{Vol}(\mathcal{D}_1)} \int_{\mathcal{D}} dw dv v^{-2} f(w, v) + O(v^{1-\frac{\Theta}{2}}) \quad v \rightarrow 0, \quad (2.5)$$

where Θ is the superior of the real part of the non trivial zeros of the Riemann zeta function, ($\Theta = \frac{1}{2}$ if and only if the Riemann hypothesis is true, while so far one can prove that $\frac{1}{2} \leq \Theta < 1$). The asymptotic (2.5) implies that an independent result on the error term would prove (or disprove) the Riemann hypothesis. The above relation has been proved to hold also for modular functions of polynomial growth for $v \rightarrow \infty$, appearing in type II string theory [ACER],[C2]. Eq. (2.5) is expected to hold also from certain arguments in heterotic strings [ACER],[C2], although it is an open challenge to prove it in this latter case [C2].

In one-loop vacua, above considerations lead to an interesting relation between ultra-violet behavior of closed string spectra and the Riemann hypothesis [ACER]. This is given by the following asymptotic:

$$\Lambda_{uv}^{d/2-1} \sum_{|\Phi\rangle} (-)^{F_\Phi} d(\Phi) e^{-\pi m_\Phi^2 / \Lambda_{uv}^2} \sim \frac{A_1}{\text{Vol}(\mathcal{D}_1)} + O(\Lambda_{uv}^{\Theta-1}) \quad \Lambda_{uv} \rightarrow \infty,$$

which implies that differences between bosonic and fermionic degrees of freedom oscillate with the frequencies given by imaginary parts of the non trivial zeros of the Riemann zeta function [ACER]. Moreover, asymptotic supersymmetry is maximal if and only if the Riemann hypothesis is true [ACER].

It is now worth to explain the Iwasawa decomposition origin of the two coordinates w and v in $\tau = w + iv$. The upper complex plane \mathcal{H}_1 is isomorphic to the Lie coset $\mathcal{H}_1 \sim Sp(2, \mathbb{R})/SO(2, \mathbb{R})$. Given a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in this coset, the bijective map is given by

$$\tau = (ai + b)(ci + d)^{-1}. \quad (2.6)$$

The Iwasawa decomposition allows to write a symplectic matrix $g \in Sp(2, \mathbb{R})$ as

$$g = \begin{pmatrix} 1 & w \\ 0 & 1 \end{pmatrix} \begin{pmatrix} v^{1/2} & 0 \\ 0 & v^{-1/2} \end{pmatrix} \begin{pmatrix} \cos \vartheta & \sin \vartheta \\ -\sin \vartheta & \cos \vartheta \end{pmatrix} \quad w \in \mathbb{R}, v > 0, \vartheta \bmod(2\pi), \quad (2.7)$$

From the isomorphism $\mathcal{H}_1 \sim Sp(2, \mathbb{R})/SO(2, \mathbb{R})$ given by the map in (2.6), one thus finds $\tau = w + iv$. Therefore, the horocycle flow along the w coordinate uplifted in the homogenous space $Sp(2, \mathbb{R})/SO(2, \mathbb{R})$ is generated by unipotent elements, given by the upper triangular matrix of the Iwasawa decomposition (2.7). On the other hand, v corresponds to the coordinate of the abelian part in the Iwasawa decomposition (2.7).

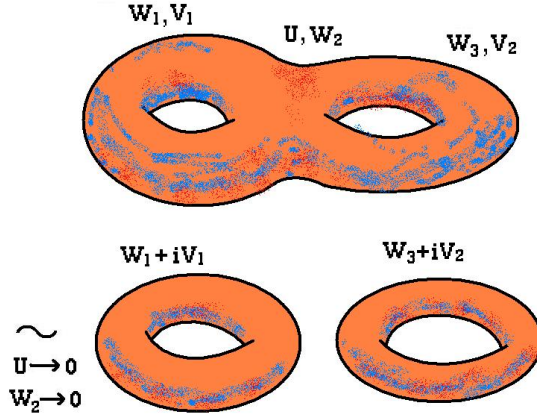


Figure 4: In the Iwasawa parametrization of the genus two Siegel half-space \mathcal{H}_2 , the degenerating limit is realized by sending to zero off-diagonal unipotent moduli [CC2].

3. Genus two: uniformization of unipotent flows viz cutting the amplitude handle(s)

In this section we illustrate in some details the genus $g = 2$ case, while higher genera are discussed in the next sections. The moduli space of genus-two compact Riemann surfaces \mathcal{M}_2 is isomorphic to $Sp(4, \mathbb{Z}) \backslash \mathcal{H}_2$. \mathcal{H}_2 is the genus two Siegel half space, given by complex symmetric two by two matrices τ , with positive definite imaginary part. \mathcal{H}_2 is isomorphic to $Sp(4, \mathbb{R}) / (SO(4, \mathbb{R}) \cap Sp(4, \mathbb{R}))$, the symplectic matrices over the orthosymplectic ones.

By Iwasawa decomposition, each element m of the above coset can be written as $m = UA$, where U is a unipotent matrix and A is a abelian matrix, (we refer to our works [CC1],[CC2] for notations and proofs [CC2] used throughout the rest of this paper).

One finds for the genus-two period matrix $\tau_{(2)}$

$$\tau_{(2)} = \begin{pmatrix} w_1 + i(v_1 + u^2 v_2) & w_2 + iuv_2 \\ w_2 + iuv_2 & w_3 + iv_2 \end{pmatrix}. \quad (3.1)$$

Thus a genus-two Riemann surface degenerates into two genus-one Riemann surfaces when both the off-diagonal unipotent moduli u and w_2 go to zero, (figure 4)

$$\tau_{(2)} = \begin{pmatrix} w_1 + i(v_1 + u^2 v_2) & w_2 + iuv_2 \\ w_2 + iuv_2 & w_3 + iv_2 \end{pmatrix} \rightarrow \begin{pmatrix} w_1 + iv_1 & 0 \\ 0 & w_3 + iv_2 \end{pmatrix} = \begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_3 \end{pmatrix}. \quad (3.2)$$

A genus-two closed string amplitude is given by a modular integral

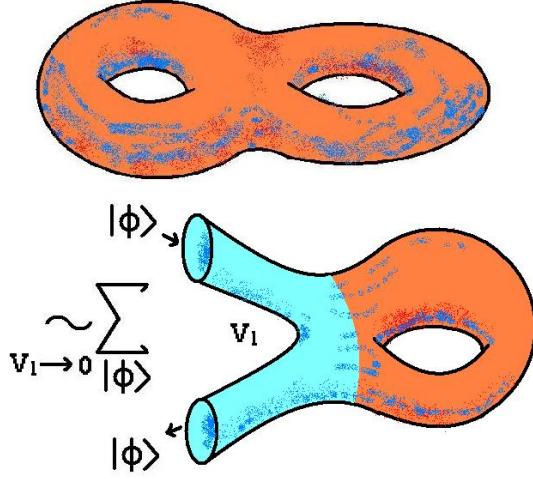


Figure 5: Uniformization theorem, (Theorem 2 in [CC2]), allows to cut an handle of a genus-two vacuum amplitude, with the prescription of summing over physical states flowing through the two marked points.

$$A_2 = \int_{\mathcal{D}_2} d\mu_2 f_2(\tau_{(2)}),$$

with $\mathcal{D}_2 \sim Sp(4, \mathbb{Z}) \backslash \mathcal{H}_2$ is a fundamental region of the genus-two modular group $\Gamma_2 \sim Sp(4, \mathbb{Z})$. Uniform distribution theorem, (Theorem 2 in [CC2]), allows to rewrite the genus-two modular integral as the integral over the corank-one component of the \mathcal{H}_2 boundary of the following f_2 unipotent average

$$A_2 = \frac{Vol(\mathcal{D}_2)}{2Vol(\mathcal{D}_1)} \lim_{v_1 \rightarrow 0} \int_{\mathcal{D}_1} d\mu_1 \int_{mod(1)} dw_1 \int dw_2 du f_2(\tau_{(2)}). \quad (3.3)$$

On the other hand, Theorem 1 in [CC2], ensures the following unipotent average function, defined on $\mathcal{H}_1 \times \mathbb{R}_{>0}$

$$f_1(\tau_3, v_1) = \int_{mod(1)} dw_1 \int dw_2 du f_2, \quad (3.4)$$

to be invariant under $SL(2, \mathbb{Z})$ modular transformations on τ_3

$$\tau_{(2)} = \begin{pmatrix} \tau_1 & \tau_2 \\ \tau_2 & \tau_3 \end{pmatrix}, \quad \tau_1 \in \mathcal{H}_1, \tau_3 \in \mathcal{H}_1.$$

From the degenerate limit (3.2), and the discussion on the genus-one case in §2, one sees that integration mod(1) along the coordinate w_1 together with the limit $v_1 \rightarrow 0$ in (3.3), diagrammatically correspond to the situation displayed in figure 5. This unipotent

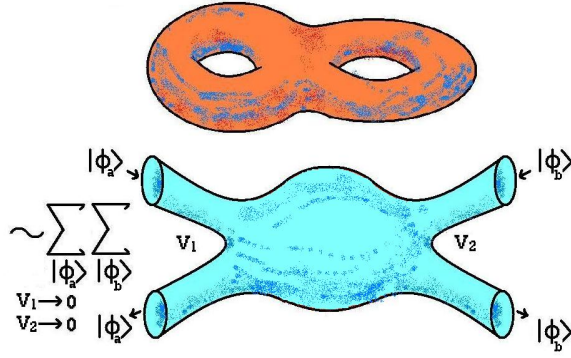


Figure 6: By applying twice the uniform distribution theorem for unipotent flows, one can write the genus two vacuum amplitude as a sum of three level four points amplitudes. This implies that on a stable vacuum, two-loop infrared finiteness is translated into a constraint on the asymptotic ultraviolet behavior of tree-level four-points closed string amplitudes.

flow representation corresponds to cutting one handle of the genus-two vacuum amplitude and replacing it by two external legs, (figure 5). This operation generates a genus-one amplitude with two external legs. Uniformization theorem [CC2] gives prescription of summing over all physical external states $|\Phi\rangle$. Therefore, infrared finiteness at genus-two order, $|A_2| < \infty$, through eq. (3.3) corresponds to finiteness of the one-loop correction to the asymptotic Fermi-Bose degeneracy condition, described in §2. Thus, we have shown that ergodicity results of unipotent flows [CC2] lead to the one-loop correction to the *asymptotic supersymmetry* constraint, obtained in [KS].

However, there is another incarnation of the genus-two vacuum amplitude, in terms of a sum of *tree-level four-point functions*, (figure 6). This is obtained by applying the uniformization theorem for unipotent flows on the $SL(2, \mathbb{Z})$ -invariant function f_1 in eq. (3.4):

$$A_2 = \frac{\text{Vol}(\mathcal{D}_2)}{2} \lim_{v_1 \rightarrow 0} \lim_{v_2 \rightarrow 0} \int_{\text{mod}(1)} dw_3 \int_{\text{mod}(1)} dw_1 \int dw_2 du f_2(\tau_{(2)}). \quad (3.5)$$

Diagrammatically, eq. (3.5) corresponds to transmute the original genu-two vacuum amplitude in a double sum of tree level four-point functions, extended to physical closed string states, (figure 6). Level matching for external states flowing in the external legs follow by integration mod(1), along the diagonal unipotent coordinates w_1 and w_3 in the period matrix (3.1). On the other hand, abelian coordinates v_1 and v_2 act as ultraviolet cutoffs for the masses of the external states.

One can also write an asymptotic expression for A_2 when v_1 and v_2 are both small. As in the genus one case described in §2, the error term turns out to be remarkably connected to the Riemann hypothesis [CC1],[CC2]:

$$A_2 \sim_{\substack{v_1 \rightarrow 0 \\ v_2 \rightarrow 0}} \frac{\text{Vol}(\mathcal{D}_2)}{2} \int_{\text{mod}(1)} dw_3 \int_{\text{mod}(1)} dw_1 \int dw_2 du f_2(\tau_{(2)}) + O(v_1^{2-\frac{\Theta}{2}}) + O(v_2^{1-\frac{\Theta}{2}}).$$

Θ is the superior of the real part of the non trivial zeros of the Riemann zeta function, ($\Theta = 1/2$ if and only if the Riemann hypothesis is true).

4. Uniformizations for closed amplitudes at genus $g = 1, 2, 3$, and moduli of punctured Riemann surfaces.

Uniformization results [CC2] concern integrals of automorphic forms on the moduli space of genus g principally polarized abelian varieties (ppav) \mathcal{A}_g . Every point in \mathcal{A}_g describes a g -dimensional torus which can be embedded in a projective space (abelian variety), with principal polarization. The moduli space of genus g compact Riemann surfaces \mathcal{M}_g is isomorphic to \mathcal{A}_g for $g = 1, 2, 3$, while for genus $g \geq 4$, \mathcal{M}_g is a subvariety fully contained in \mathcal{A}_g of (complex) codimension $\dim(\mathcal{A}_g) - \dim(\mathcal{M}_g) = \frac{1}{2}(g-2)(g-3)$.

Uniformization results for modular integrals of automorphic forms on \mathcal{A}_g [CC2], when applied to closed string vacuum amplitudes connect a genus g vacuum amplitude to sums of lower genera scattering amplitudes. Diagrammatically, one transmutes a handle in the Riemann surface into a pair of marked points. For each cut handle, the amplitude genus g is lowered by a unit, while two external legs are added to the amplitude. For the moduli space of genus g compact Riemann surfaces with n marked points $\mathcal{M}_{g,n}$, one has $\dim(\mathcal{M}_{0,n}) = n - 3$ (Riemann sphere with n marked points), $\dim(\mathcal{M}_{1,n}) = n$ (torus with n marked points), $\dim(\mathcal{M}_{g,n}) = 3g - 3 + n$, $g \geq 2$.

For the genus one-amplitude (torus) discussed in §2, the uniform distribution of long horocycles theorem reduces this amplitude in a sum of sphere amplitudes with two marked points, (figure 3). In the original torus modulus $\tau = w + iv$, integration mod(1) of the unipotent Iwasawa coordinate w forces level matching for the external closed string states flowing through the two marked points. The abelian Iwasawa coordinate v provides a UV cutoff $\Lambda = 1/v$ for their masses. There are no extra moduli besides w and v , consistently with the absence of moduli for Riemann spheres with two marked points, $\dim(\mathcal{M}_{0,2}) = 0$.

For the genus-two vacuum amplitude discussed in §3, uniformization results [CC2] give two representations. One is given as a sum of genus-one amplitudes with two marked points, (figure 5), while the other one is given in terms of a sum of genus-zero amplitudes with four marked points, (figure 6). The first case is displayed in eq. (3.3), with integrated moduli w_1 , w_2 , u and $w_3 + iv_2$ in the periods matrix $\tau_{(2)}$, eq. (3.1). Integration on w_1 mod(1) ensures level matching for the closed string states flowing through the two marked points, while $w_3 + iv_2$ is the modulus related to the left over handle. $w_3 + iv_2$ together with w_2 and u consistently account for the number of moduli of tori with two marked points, $\dim(\mathcal{M}_{1,2}) = 2$, (complex dimension). For the second representation of A_2 , displayed in eq. (3.5), the genus-two vacuum amplitude is given by a sum over genus-zero amplitudes

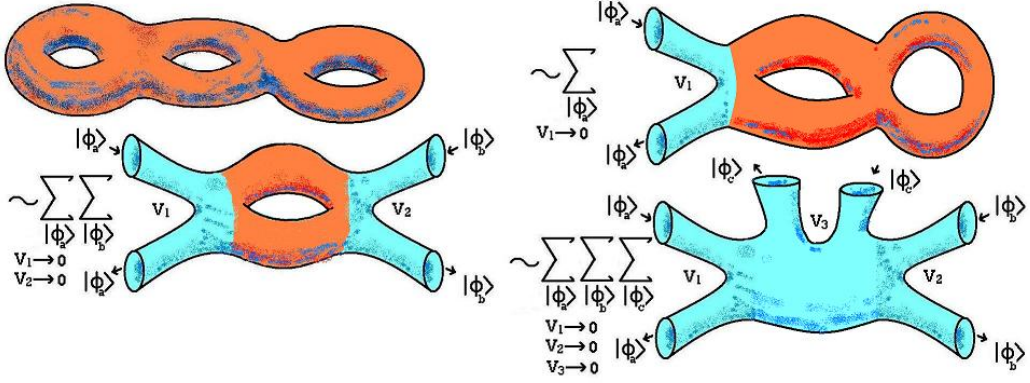


Figure 7: Three different representations for a genus three closed string vacuum amplitude, that follow from the uniformization theorems of unipotent flows [CC2].

with four marked points. The two diagonal unipotent moduli w_1 and w_3 integrated mod(1) ensure level matching for states flowing through the two pairs of marked points, (figure 6). On the other hand, the off-diagonal unipotent moduli w_2 and u consistently account for the dimension of the moduli space of spheres with four marked points $\dim(\mathcal{M}_{0,4}) = 1$.

At genus three, the periods matrix in Iwasawa parametrization of \mathcal{H}_3 is given by

$$\begin{aligned} \tau_{(3)} &= \begin{pmatrix} w_{11} + i(v_1 + v_2 u_{12}^2 + v_3 u_{13}^2) & w_{12} + i(v_2 u_{12} + v_3 u_{12} u_{13}) & w_{13} + i v_3 u_{13} \\ * & w_{22} + i(v_2 + v_3 u_{23}^2) & w_{23} + i v_3 u_{23} \\ * & * & w_{33} + i v_3 \end{pmatrix} \\ &= \begin{pmatrix} \tau_{11} & \tau_{12} \\ \tau_{12}^t & \tau_{22} \end{pmatrix}, \quad \tau_{11} \in \mathcal{H}_1, \tau_{22} \in \mathcal{H}_2. \end{aligned}$$

where $*$ entries are given by symmetry. Uniformization theorems in [CC2] give three alternative representations for the closed string vacuum amplitude A_3 , (figure 7)

By cutting one handle, one can write A_3 as a sum of genus-two amplitudes with two marked points, while by cutting two handles A_3 is given by a sum of genus-one amplitudes with four marked points. Finally, by cutting three handles A_3 can be expressed as a sum of genus-zero amplitudes with six marked points.

The representation in terms of genus-two amplitudes is given by

$$A_3 \sim_{v_1 \rightarrow 0} \frac{\text{Vol}(\mathcal{D}_3)}{2\text{Vol}(\mathcal{D}_2)} \int_{\mathcal{D}_2} d\mu_2 \int_{\text{mod}(1)} dw_{11} \int dw_{12} dw_{13} du_{12} du_{13} f_3(\tau_{(3)}).$$

The diagonal unipotent modulus w_{11} integrated mod(1) enforces level matching for the states flowing thorough the two marked points. The off-diagonal unipotent moduli $w_{12}, w_{13}, u_{12}, u_{13}$, together with $\tau_{22} \in \mathcal{H}_2$, (the periods matrix of the two-leftover handles),

account for the dimension of the moduli space of genus-two Riemann surfaces with two marked points $\dim(\mathcal{M}_{2,2}) = 5$.

The representation of A_3 as a sum of genus-one amplitudes with four marked points is given by

$$A_3 \sim_{\substack{v_1 \rightarrow 0 \\ v_2 \rightarrow 0}} \frac{\text{Vol}(\mathcal{D}_3)}{4\text{Vol}(\mathcal{D}_1)} \int_{\mathcal{D}_1} d\mu_1 \int_{\text{mod}(1)} dw_{11} \int_{\text{mod}(1)} dw_{22} \int dw_{12} dw_{13} dw_{23} du_{12} du_{13} du_{23} f_3(\tau_{(3)}),$$

Diagonal unipotent moduli w_{11} and w_{22} integrated mod(1) select physical states through the two pairs of marked points, (figure 8). Off-diagonal unipotent coordinates $w_{12}, w_{13}, w_{23}, u_{12}, u_{13}, u_{23}$ and $\tau_{(1)} \in \mathcal{H}_1$ correctly account for the dimension of the moduli space of tori with four marked points, $\dim(\mathcal{M}_{1,4}) = 4$.

The representation of A_3 as a sum of genus-zero amplitudes with six marked points is given by

$$A_3 \sim_{\substack{v_1 \rightarrow 0 \\ v_2 \rightarrow 0 \\ v_3 \rightarrow 0}} \frac{\text{Vol}(\mathcal{D}_3)}{8} \int_{\text{mod}(1)} dw_{11} \int_{\text{mod}(1)} dw_{22} \int_{\text{mod}(1)} dw_{33} \int dw_{12} dw_{13} dw_{23} du_{12} du_{13} du_{23} f_3(\tau_{(3)}).$$

Diagonal unipotent moduli w_{11}, w_{22}, w_{33} integrated mod(1) select physical closed string states through the three pairs of marked points (figure 7). Off-diagonal unipotent moduli $w_{12}, w_{13}, w_{23}, u_{12}, u_{13}, u_{23}$ consistently account for the dimension of the moduli space of spheres with six marked points, $\dim(\mathcal{M}_{0,6}) = 3$.

5. Uniformization at every genera: closed string hints for relevance of unipotent flows for the Schottky problem.

Uniformization results [CC2] concern integrals of $Sp(2g, \mathbb{Z})$ automorphic forms on the moduli space of genus g principally polarized abelian varieties \mathcal{A}_g . The moduli space of genus g compact Riemann surfaces \mathcal{M}_g is isomorphic to \mathcal{A}_g for $g = 1, 2, 3$. For genus $g \geq 4$, \mathcal{M}_g is a subvariety fully contained in \mathcal{A}_g , of (complex) codimension $\dim(\mathcal{A}_g) - \dim(\mathcal{M}_g) = \frac{1}{2}(g-2)(g-3)$. The embedding of \mathcal{M}_g in \mathcal{A}_g is called the Schottky locus \mathcal{S}_g , and the problem of its complete characterization for every genus is still wide open, (see for example [Gr]). Genus four is the first non-trivial case, and the Schottky locus \mathcal{S}_4 is a divisor in \mathcal{A}_4 , (it is of complex codimension one). In this case, \mathcal{S}_4 is fully characterized by the vanishing of a 16-degree polynomial in the theta nulls, (the Igusa form I_4).

From a string theory diagrammatic perspective, it seems that nothing special occurs at genus $g = 4$, or higher. This leads us to conjecture for the cutting-handles procedure to hold at every genera

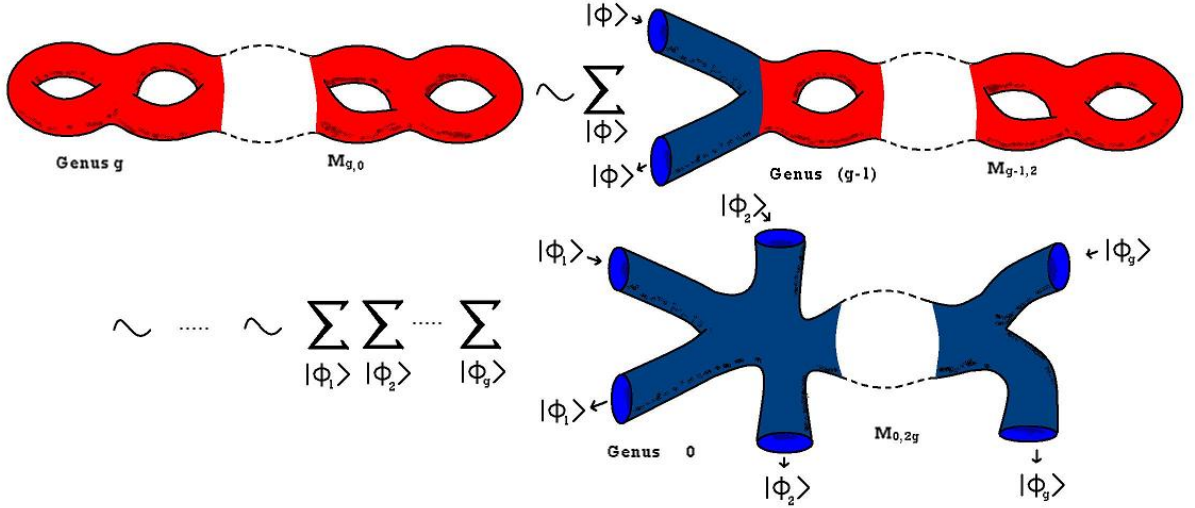


Figure 8: Uniformization relations for a vacuum amplitude of arbitrary genus.

$$\mathcal{A}_{g,0} = \sum_{|\Phi\rangle} \mathcal{A}_{g-1,2} = \cdots = \sum_{a=1}^g \sum_{|\Phi_a\rangle} \mathcal{A}_{0,2g}(|\Phi_1\rangle, \dots, |\Phi_g\rangle),$$

where

$$\mathcal{A}_{g,2n} = \int_{\mathcal{M}_{g,2n}} d\mu_g f_{g,n}(|\Phi_1\rangle, \dots, |\Phi_n\rangle),$$

is the $2n$ -points genus g closed string amplitude with external states $|\Phi_1\rangle, \dots, |\Phi_n\rangle$ arranged as in figure 8.

This suggests the interesting possibility that results in the dynamics of unipotent flows may be used to treating modular integrals restricted to the Schottky locus \mathcal{S}_g . This is of interest for the still open problem of constructing closed string amplitudes of arbitrary genus. Moreover, one may use unipotent flows to connect recently proposed genus three [CDPvG], (and genus higher than three [CDPvG], [Gr2], [GSM1],[GSM2],[MV1], [MV2], [MV3]), closed string amplitudes to the genus-two expressions given in [DHP]. All the above possibilities are presently under investigation [CC3], [CCDP].

6. Summary and Future Directions

In this paper we have applied to string theory mathematical results on uniform distribution of unipotent flows we have recently proved [CC2]. We obtained conditions of perturbative stability *at all genera* in non-supersymmetric closed string vacua, in terms of *solely* closed string three-level diagrams. The key is provided by ergodic properties of unipotent flows,

which allow to study closed strings asymptotic UV properties, by probing boundary components of \mathcal{M}_g , (the genus g moduli space of compact Riemann surfaces). Diagrammatically, unipotent flows asymptotics translate into prescriptions of cutting amplitudes handles, while summing over physical states flowing through pairs of marked points. Remarkably, those asymptotics have error estimates related to the Riemann hypothesis [CC1],[CC2].

Our analysis extends to *all genera* in perturbation theory, and generalizes previous results at genus-one level [KS], (see also [Di]). It would be interesting to check whether our condition on ultraviolet perturbative stability of a closed string vacuum make contact with stability conditions in closed string field theory. Moreover, our closed string theory constraints may be of interest for higher spins theories, (see for example [FS],[ST]). We also notice some formal similarities between our closed string cutting procedures and analytic cuts procedures in quantum field theory [Dix]. It would be interesting to check whether those analogies point to some deep relations. We also would like to mention that our cutting techniques seem to suit for connecting loops vacuum closed string amplitudes to the most general closed string tree-level amplitudes. This may be achieved by taking appropriate limits for vertex operators on the punctured Riemann surface [Sa]. Progress in this direction, would provide explicit expressions in type II A, type II B and Type 0 closed string theories for genus two and genus three closed string amplitudes [Sa].

From the mathematical side, our diagrammatic prescriptions on closed string amplitudes suggest a relevance of unipotent flows for the Schottky problem, and for the problem of defining the superstring amplitude at arbitrary genera. These issues are presently under investigation [CC3], [CCDP], in two distinct directions. On the one hand, we would like to obtain uniformization results for integrals of automorphic functions over the Schottky locus. Existence of such an interesting possibility is suggested diagrammatically by string theory [CC3]. On the other hand, by using uniformization result in [CC2], we would like to be able to connect recently proposed genus $g = 3, 4, 5$ closed string amplitudes [CDPvG],[DPvG],[DPGC], [Gr],[GSM1],[GSM2],[Mo],[MV1],[MV2],[MV2] to the genus two superstring amplitudes given in [DHP]. This would provide consistency checks for the proposed genus $g \geq 3$ superstring amplitudes [CCDP].

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